ROTATION FIELDS

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Introduction. Tom Wieting spoke today on "Rainich's Theory Revisited" at a physics seminar. Conversation at dinner touched upon a question that has haunted me for nearly forty years: What can one say about the structure of the "Lorentz field" $\mathbb{\Lambda}(x)$ that results from exponentiation of the electromagnetic field matrix $\mathbb{F}(x)$? On these pages I explore an idea that occurred to me while driving home from that event.

2-dimensional rotation fields. Let

$$\mathbb{R}(x^1,x^2) = e^{\lambda\mathbb{A}} \quad\text{with}\quad \mathbb{A} = \theta(x^1,x^2)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
$$\equiv \theta(x)\,\mathbb{J}$$

where λ is intended to play the role of a "dimensionally enforced constant," analogous to the $\hbar$ in

$$\psi = Re^{\frac{i}{\hbar}S}$$

Then

$$\mathbb{R}(x) = e^{\lambda\theta(x)\,\mathbb{J}} = \begin{pmatrix} \cos\lambda\theta(x) & -\sin\lambda\theta(x) \\ \sin\lambda\theta(x) & \cos\lambda\theta(x) \end{pmatrix}$$

and we have

$$\begin{aligned}
\partial_i\,\mathbb{R} \equiv \mathbb{R}_i &= \lambda\begin{pmatrix} -\sin\lambda\theta & -\cos\lambda\theta \\ \cos\lambda\theta & -\sin\lambda\theta \end{pmatrix}\cdot\theta_i \\
&= \lambda\,\mathbb{J}\,\mathbb{R}\cdot\theta_i \\
\mathbb{R}_{ij} &= \lambda\,\mathbb{J}\,\mathbb{R}_j\cdot\theta_i + \lambda\,\mathbb{J}\,\mathbb{R}\cdot\theta_{ij} \\
&= \lambda\,\mathbb{J}\,(\lambda\,\mathbb{J}\,\mathbb{R}\cdot\theta_j)\cdot\theta_i + \lambda\,\mathbb{J}\,\mathbb{R}\cdot\theta_{ij} \\
&= -\lambda^2\,\mathbb{R}\cdot\theta_i\theta_j + \lambda\,\mathbb{J}\,\mathbb{R}\cdot\theta_{ij} \qquad\qquad \text{by}\quad \mathbb{J}^2 = -\mathbb{I} \\
\mathbb{R}_{ijk} &= -\lambda^2\,(\lambda\,\mathbb{J}\,\mathbb{R}\cdot\theta_k)\cdot\theta_i\theta_j - \lambda^2\,\mathbb{R}\cdot\theta_{ik}\theta_j - \lambda^2\,\mathbb{R}\cdot\theta_i\theta_{jk} \\
&\qquad\qquad + \lambda\,\mathbb{J}\,(\lambda\,\mathbb{J}\,\mathbb{R}\cdot\theta_k)\cdot\theta_{ij} + \lambda\,\mathbb{J}\,\mathbb{R}\cdot\theta_{ijk} \\
&= -\lambda^3\,\mathbb{J}\,\mathbb{R}\cdot\theta_i\theta_j\theta_k - \lambda^2\,\mathbb{R}\cdot(\theta_{ij}\theta_k + \theta_{jk}\theta_i + \theta_{ki}\theta_j) + \lambda\,\mathbb{J}\,\mathbb{R}\cdot\theta_{ijk} \\
&\;\;\vdots
\end{aligned}$$

Suppose now that it were, for example, the case that physics supplied

$$g^{ij}\theta_i\theta_j + f = 0 \tag{1}$$

From

$$\theta_i\theta_j\mathbb{I} = -\lambda^{-2}\,\mathbb{R}^{-1}\Big\{\mathbb{R}_{ij} - \lambda\,\mathbb{J}\,\mathbb{R}\cdot\theta_{ij}\Big\}$$

we obtain

$$-\lambda^{-2}\,\mathbb{R}^{-1}\Big\{g^{ij}\,\mathbb{R}_{ij} - \lambda\,\mathbb{J}\,\mathbb{R}\cdot g^{ij}\theta_{ij}\Big\} + f\,\mathbb{I} = \mathbb{O}$$

or

$$\lambda^{-2}\Big\{g^{ij}\,\mathbb{R}_{ij} - \lambda\,\mathbb{R}^{-1}\mathbb{J}\,\mathbb{R}\cdot g^{ij}\theta_{ij}\Big\} - f\,\mathbb{R} = \mathbb{O}$$

If λ is "small" (other assumptions would serve the same purpose) the second term in $\{$etc.$\}$ drops away, and we are left with an equation

$$\big\{\nabla^2 - \lambda^2 f\big\}\mathbb{R} = \mathbb{O} \tag{2}$$

which is *linear* in $\mathbb{R}$. Equation (1) would acquire then the status of an "eikonal approximation" to the Helmholtz equation (2).

Alternatives to (1) could in some cases be managed similarly. Let us, with some kind of toy electrodynamics in mind, suppose it to be known that

$$\partial_i A^i{}_j = C_j \tag{3}$$

i.e., that

$$\begin{aligned}\partial_1 A^1{}_1 + \partial_2 A^2{}_1 &= C_1\\ \partial_1 A^1{}_2 + \partial_2 A^2{}_2 &= C_2\end{aligned}$$

Then

$$\partial_2\theta = C_1 \quad\text{and}\quad -\partial_1\theta = C_2$$

impose upon $\boldsymbol{C}$ the condition

$$\partial_1 C_1 + \partial_2 C_2 = 0 \quad ;\textit{i.e.,}\quad \boldsymbol{\nabla}\!\cdot\!\boldsymbol{C} = 0$$

and in the trivial case $\boldsymbol{C} = \boldsymbol{0}$ we are forced to set

$$\theta(x) = \text{constant} \quad : \quad \theta_1 = \theta_2 = 0$$

Then so is $\mathbb{R}(x)$ constant: $\mathbb{R}_i = \mathbb{R}_{ij} = \mathbb{R}_{ijk} = \cdots = \mathbb{O}$. Not very interesting. But if $\boldsymbol{C} \neq \boldsymbol{0}$ we have

$$\begin{aligned}\begin{pmatrix}\partial_1\\ \partial_2\end{pmatrix}\mathbb{R} &= \lambda\,\mathbb{J}\,\mathbb{R}\begin{pmatrix}-C_2\\ +C_1\end{pmatrix}\\ &= \lambda\,\mathbb{J}\,\mathbb{R}\,\mathbb{J}\begin{pmatrix}C_1\\ C_2\end{pmatrix}\\ &= -\lambda\,\mathbb{R}\begin{pmatrix}C_1\\ C_2\end{pmatrix} \quad\Longrightarrow\quad \big\{\partial_i + \lambda C_i\big\}\mathbb{R} = \mathbb{O}\end{aligned} \tag{4}$$

This relatively more interesting equation gives back $\mathbb{R}_i = \mathbb{O}$ when $\boldsymbol{C} = \mathbf{0}$, and has been obtained without approximation (no step comparable to the assumption that "λ is small").

2-dimensional Lorentz fields. One proceeds from

$$\begin{aligned}\mathbb{\Lambda}(x^0, x^1) = e^{\lambda\mathbb{B}} \quad \text{with} \quad \mathbb{B} &= \psi(x^0, x^1)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &\equiv \psi(x)\,\mathbb{K}\end{aligned}$$

Details change, but all essential ideas remain intact.

3-dimensional rotation fields. One proceeds from